*Discipline:* **Physics**
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*Unit 18:*
*Lesson/ Module:* **Lagrangian of a Charged Particle**

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***Learning Objectives:***

***From this module students may get to know about the following:***

1. *Lagrangian and Hamiltonian formalism in Newtonian mechanics.*
2. *Lagrangian formulation for relativistic mechanics and the Lagrangian of a relativistic particle.*
3. *Lagrangian of a relativistic charged particle in the presence of electric and magnetic fields but in a non-covariant treatment.*
4. *The Hamiltonian of a relativistic charged particle in the presence of electric and magnetic fields but in a non-covariant treatment.*
5. *The manifestly covariant treatment of the Lagrangian and Hamiltonian formulation.*

# I8. Lagrangian and Hamiltonian formulation for a charged particle

## *18.1 Lagrangian and Hamiltonian in non-relativistic mechanics*

Although the equations of motion are sufficient to describe the motion of a charged particle in external electromagnetic fields, it is useful to consider the formulation of dynamics from the Lagrangian and Hamiltonian points of view. We begin with a very short review of the Lagrangian formulation of classical mechanics. In a certain sense Lagrangian mechanics is only a reformulation of Newtonian mechanics. From a practical point of view, whereas in Newtonian mechanics we deal with forces which are vector quantities, in Lagrangian or Hamiltonian formulation we deal with a single scalar function of the "generalized" coordinates, and velocities or momenta. Newtonian mechanics works very nicely in Cartesian coordinates, but it is difficult to switch to a different coordinate system. Something as simple as changing to polar coordinates is cumbersome; finding the equations of motion of a particle acting under a "central force" in polar coordinates is tedious. The Lagrangian formulation, in contrast, is *independent* of the coordinates, and the equations of motion for a non-Cartesian coordinate system can typically be found immediately on using it.

However, much more importantly, it puts the foundations of mechanics on very broad basis from which an extension to other physical systems becomes clear; or you can say it brings out the connection with other physical systems – quantum mechanics, statistical mechanics electrodynamics, field theory and so on, and emphasizes the uniformity of the underlying principles and basis of physical phenomena.

The Lagrangian treatment is based on the principle of least action or Hamilton's principle. In nonrelativistic mechanics the system is described by a set of *n* generalized coordinates $q_i(t)$ and velocities $\dot{q}_i(t)$, *n* being the number of degrees of freedom. The Lagrangian is a scalar *functional* (in ordinary three dimensional space) of $q_i(t)$, $\dot{q}_i(t)$ and perhaps explicitly of time as well. The *action A* is defined as the time-integral of the Lagrangian *L* along a possible path of a system. The *principle of least action* states that the motion of the system is such that in going from a configuration *a* at time $t_1$ to configuration *b* at time $t_2$, the action

$$A = \int_{t_1}^{t_2} L[q_i(t), \dot{q}_i(t), t] dt \qquad (1)$$

is an extremum. By considering small variations of the coordinates and velocities away from the actual path and requiring $\delta A = 0$, the condition for the extremum, we obtain the Euler-Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0; \qquad i = 1,2,..n \qquad (2)$$

Given the Lagrangian $L(q_i, \dot{q}_i, t)$ of a system, we can define the Hamiltonian *H*. Whereas the Lagrangian is a function of generalized coordinates and generalized velocities, the

Hamiltonian is to be regarded as a function of generalized coordinates and generalized momenta: $H(q_i, p_i, t)$. The generalized momenta are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{3}$$

The Hamiltonian is defined in terms of the Lagrangian as

$$H(q_i, p_i, t) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q_i, \dot{q}_i, t) \tag{4}$$

The generalized velocities of course have to be eliminated by the use of equation (3), which, in principle, can be solved for $q_i$ in terms of $p_i$. The Hamiltonian equations of motion, which are completely equivalent to the Euler-Lagrange equations, and can be derived from them, are

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i; \qquad \frac{\partial H}{\partial p_i} = \dot{q}_i \tag{5}$$

Let us apply these equations to some simple cases of nonrelativistic classical mechanics;

- If $T$ is the kinetic energy of a particle and its Lagrangian is given by,

$$L(x, \dot{x}, t) = T = \frac{1}{2} m \dot{x}^2 \Rightarrow \frac{\partial L}{\partial \dot{x}} = m\dot{x}; \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} = 0$$

This is the equation of a free particle moving along the $x$-direction. The generalized momentum is $p = m\dot{x}$, the same as ordinary momentum and the Hamiltonian is $H = \frac{p^2}{2m} = T$, the energy of the particle.

- If $T$ is the kinetic energy and $V$ the potential energy of a particle and

$$L(\vec{x}, \dot{\vec{x}}, t) = T - V(\vec{x}) = \frac{1}{2} m \dot{x}^2 - V(\vec{x}) \Rightarrow \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}};$$
$$\frac{\partial L}{\partial \vec{x}} = -\frac{\partial V}{\partial \vec{x}} \Rightarrow m\ddot{\vec{x}} = -\frac{\partial V}{\partial \vec{x}} = F(\vec{x})$$

This is the equation of motion of a particle moving under the action of a force $\vec{F}(\vec{x})$ represented by the potential $V(\vec{x})$. The generalized momentum is $\vec{p} = m\dot{\vec{x}}$, the same as ordinary momentum and the Hamiltonian is $H = \frac{p^2}{2m} + V(\vec{x}) = T + V$, the total energy.

### *18.2 Lagrangian for relativistic motion of a free particle*

We wish to now extend this formalism to the relativistic motion of a charged particle moving under the action of external electromagnetic fields and consistent with the requirements of special theory of relativity. Let us first present the straightforward treatment of the problem in which we continue with the ordinary coordinates and velocities but generalize to the nonrelativistic domain. At the end of this module we will present the more sophisticated covariant treatment.

Let us first consider the question of the Lorentz transformation properties of the Lagrangian. From the first postulate of special relativity the action integral must be a Lorentz scalar because the equations of motion are determined by the extremum condition, $\delta A = 0$, and they must be the same in all frames of reference. Or we may think of invariance of *A* as an assumption or postulate in its own right and see where it leads us. If we introduce particle's proper time $\tau$ in equation (1) through $dt = \gamma d\tau$, where $\gamma = 1/\sqrt{1-u^2/c^2}$ and *u* the velocity of the particle, the action integral becomes

$$A = \int_{\tau_1}^{\tau_2} \gamma L \, d\tau \qquad (6)$$

Thus if we believe that *A* is Lorentz invariant then so is $\gamma L$. This statement of invariance greatly limits possible forms of the Lagrangian.

Let us first consider the case of a free particle. What invariants we may construct from properties of a free particle? We have only the momentum four-vector $p^\mu$ and the position four-vector $x^\mu$. The presumed translational invariance of space rules out the use of the later. That leaves only the momentum four vector $p^\mu$ and the single invariant $p^\mu p_\mu = m^2c^2$, which is a constant. Hence we are led to $\gamma L = C$, a constant. To find the constant *C*, we appeal to the nonrelativistic limit and expand in powers of *u/c*:

$$L = C/\gamma = C\sqrt{1-\frac{u^2}{c^2}} = C(1-\frac{1}{2}\frac{u^2}{c^2}+\ldots)$$

On comparing with the Lagrangian $\frac{1}{2}mu^2$ for the nonrelativistic case, we have $C = -mc^2$ (the first term of the series is *C* a constant and an arbitrary constant can always be added to the Lagrangian without changing anything), and therefore

$$L_{free} = -mc^2\sqrt{1-\frac{u^2}{c^2}}\,. \qquad (7)$$

On applying the Euler-Lagrange equation (2) to the above Lagrangian we obtain the equation of motion of a free particle:

$$\frac{d}{dt}(m\gamma\vec{u}) = 0\,;\text{ or } \frac{d}{dt}(\vec{p}) = 0\,. \qquad (8)$$

The action, equation (6), is proportional to the integral of the proper time over the path from the initial proper time $\tau_1$ to the final proper time $\tau_2$. This integral is Lorentz invariant but depends on the path taken from initial to final configuration. For purposes of calculation consider a frame of reference in which the particle is initially at rest. From the definition of the proper time, $d\tau = dt\sqrt{1-\frac{u^2}{c^2}}$ , it is clear that, if the particle stays at rest in that frame, the integral over proper time will be larger than if it moves with a nonzero velocity along its path. Consequently, we see that a straight *world line* joining the initial and the final configuration of the system gives the maximum value for the integral (6) over the proper time. And with the negative sign for Lagrangian $L$ from equation (7) it gives a minimum for the action integral of equation (6). This motion at constant velocity is, of course, the solution of the free particle equation of motion (8).

### *18.3 Lagrangian for a charged particle in electromagnetic field*

Now we come to the problem of finding the Lagrangian for the motion of a charged particle in an external electromagnetic field. For a particle of mass *m*, charge *q* and velocity $\vec{u}$ moving in external fields $\vec{E}$ and $\vec{B}$, the equation of motion is the Lorentz force equation

$$\frac{d\vec{p}}{dt} = q[\vec{E} + \vec{u}\times\vec{B}] \qquad (9)$$

We also have the energy change equation

$$\frac{dW}{dt} = q\vec{u}.\vec{E}\ . \qquad (10)$$

The magnetic force being perpendicular to the velocity does no work. We have used the symbol *W* for energy not to confuse it with the electric field. Together, these two equations can be put in a covariant form

$$\frac{dU^\alpha}{d\tau} = \frac{q}{m}F^{\alpha\beta}U_\beta\ . \qquad (11)$$

Here $U^\alpha = (\gamma c, \gamma\vec{u})$ is the four-velocity. $F^{\alpha\beta}$ is the field four-tensor:

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \qquad (12)$$

The general requirement that $\gamma L$ be Lorentz invariant allows us to determine the Lagrangian for a relativistic charged particle in external electromagnetic fields provided we know something about the Lagrangian or the equations of motion for the nonrelativistic motion of the particle. The total Lagrangian is $L = L_{free} + L_{\text{int}}$, where $L_{int}$ is the interaction

Lagrangian, and contains the information about fields and forces. For the interaction to be invariant, it must be the case that

$$A_{\text{int}} = \int_{t_1}^{t_2} L_{\text{int}} dt \tag{13}$$

is an invariant, which implies that ($\gamma\, L_{int}$ ) is an invariant. Now in the nonrelativistic limit we have $L$=$T$-$V$. A slowly moving charged particle is influenced predominantly by the electric field. The potential energy of the particle to lowest order is $V = q\Phi$, where $\Phi$ is the scalar potential. So we have in this limit $\gamma L_{\text{int}} = -\gamma q\Phi = -q\Phi \dfrac{W}{mc^2} = -\dfrac{q}{m} p_0 A^0$. The Lorentz invariant expression for $\gamma L_{\text{int}}$ that reduces to this form in the nonrelativistic limit is

$$\gamma L_{\text{int}} = -\frac{q}{m} p_\alpha A^\alpha = -\frac{q}{m}(\gamma m\Phi - \gamma m\vec{u}.\vec{A})$$

Thus

$$L_{\text{int}} = -\frac{q}{\gamma m} p_\alpha A^\alpha = -q\Phi + q\vec{u}.\vec{A} \tag{14}$$

This choice of interaction Lagrangian reduces to the correct static limit. It is the simplest choice for the interaction Lagrangian with the following properties:

- Possesses translational invariance (in the sense that it does not depend on $\vec{x}$ explicitly; the potentials do depend on it).
- Is linear in charge, momentum and fields.
- Does not involve time derivatives of $p^\alpha$.

Combining the two terms, $L_{free}$ and $L_{\text{int}}$ the total relativistic Lagrangian for a charged particle is

$$L = L_{free} + L_{\text{int}} = -mc^2\sqrt{1-\frac{u^2}{c^2}} - q\Phi + q\vec{u}.\vec{A} \tag{15}$$

We can now find the equations of motion by using the Euler-Lagrange equations (2). Let us calculate the various terms involved:

$$\frac{\partial L}{\partial x_i} = -q\frac{\partial \Phi}{\partial x_i} + q\frac{\partial}{\partial x_i}(\vec{u}.\vec{A})$$

$$\frac{\partial L}{\partial u_i} = qA_i - (mc^2)(-\frac{u_i}{c^2})(1-\frac{u^2}{c^2})^{-1/2} = qA_i + m\gamma u_i$$

$$\frac{d}{dt}\frac{\partial L}{\partial u_i} = q\frac{dA_i}{dt} + \frac{dp_i}{dt}$$

Now $A_i$ is the vector potential of the particle and depends on its position $\vec{x}$, which itself is a function of time $t$. So

$$\frac{dA_i(\vec{x}(t),t)}{dt} = \frac{\partial A_i}{\partial t} + \frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt} = \frac{\partial A_i}{\partial t} + (\vec{u}.\vec{\nabla})A_i$$

Putting all these into equation (2), we have

$$\frac{d}{dt}\frac{\partial L}{\partial u_i} - \frac{\partial L}{\partial x_i} = q[\frac{\partial A_i}{\partial t} + (\vec{u}.\vec{\nabla})A_i] + \frac{dp_i}{dt} + q\frac{\partial \Phi}{\partial x_i} - q\frac{\partial}{\partial x_i}(\vec{u}.\vec{A}) = 0$$

or

$$\frac{d\vec{p}}{dt} = -q[\frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\Phi] + q[\vec{\nabla}(\vec{u}.\vec{A}) - (\vec{u}.\vec{\nabla})\vec{A}]$$

The first term on the right hand side is nothing but the electric field:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\Phi \qquad (16)$$

The second term is $\vec{u}\times\vec{B}$:

$$\vec{u}\times\vec{B} = \vec{u}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{u}.\vec{A}) - (\vec{u}.\vec{\nabla})\vec{A} \qquad (17)$$

Thus finally we have the required equation of motion

$$\frac{d\vec{p}}{dt} = q[\vec{E} + \vec{u}\times\vec{B}]$$

### *18.4 Hamiltonian for a charged particle*

We next give a Hamiltonian description of the system. First we define the *canonical momentum* $\vec{P}$ *conjugate* to the position coordinate of the particle $\vec{x}$ by

$$P_i \equiv \frac{\partial L}{\partial u_i} = \gamma m u_i + qA_i \qquad (18)$$

or

$$\vec{P} = \vec{p} + q\vec{A} \qquad (19)$$

where $\vec{p} = \gamma m\vec{u}$ is the ordinary kinetic momentum of the particle. The Hamiltonian $H$ is a function of the coordinate $\vec{x}$ and its conjugate momentum $\vec{P}$ and is defined in terms of the Lagrangian as

$$H = \vec{P}.\vec{u} - L \qquad (20)$$

Since the Lagrangian is a function of $\vec{x}$ and $\vec{u}$, but Hamiltonian is a function of $\vec{x}$ and $\vec{P}$, the velocity $\vec{u}$ must be eliminated from the above expression (20). From equation (18) or (19) we obtain

$$p^2 = (\vec{P} - q\vec{A})^2 = \frac{m^2u^2}{1 - \frac{u^2}{c^2}} \Rightarrow u^2 = \frac{c^2p^2}{p^2 + m^2c^2} \Rightarrow \gamma = \frac{\sqrt{p^2 + m^2c^2}}{mc}$$

Substituting for $\vec{p}$ from (19) and for $\gamma$ from above into equation (18), we obtain

$$\vec{u} = \frac{c(\vec{P} - q\vec{A})}{\sqrt{\left(\vec{P} - q\vec{A}\right)^2 + m^2c^2}} \qquad (18)$$

When this is substituted into equation (17), the Hamiltonian takes on the form

$$H(\vec{x}, \vec{P}) = c\sqrt{(\vec{P} - q\vec{A}) + m^2c^2} + q\Phi\,. \qquad (21)$$

In the Hamiltonian formulation, the equations of motion are

$$\frac{\partial H}{\partial \vec{P}} = \dot{\vec{x}}; \qquad \frac{\partial H}{\partial \vec{x}} = -\dot{\vec{P}} \qquad (22)$$

Equation (21) is an expression for the total energy $W$ of the particle. It differs from the free-particle energy by the addition of potential energy $e\Phi$ and by the replacement $\vec{p} \to [\vec{P} - q\vec{A}]$. These two modifications are actually only one four-vector change. This can be seen by transposing $e\Phi$ to the left and squaring:

$$(W - q\Phi)^2 - c^2(\vec{P} - q\vec{A})^2 = (mc^2)^2 \qquad (23)$$

This is just the four vector scalar product,

$$p_\alpha p^\alpha = m^2c^2, \qquad (24)$$

where

$$p^\alpha \equiv (\frac{E}{c}, \vec{p}) = (\frac{1}{c}(W - q\Phi), \vec{P} - q\vec{A}) \qquad (25)$$

We see that the total energy $W/c$ acts as the time component of a canonically conjugate four-momentum $P^\alpha$ of which $\vec{P}$ given by equation (19) is the space part. A manifestly covariant approach leads naturally to this four-momentum.

What about the gauge transformation properties of the Lagrangian. The Lagrangian being a function of the potentials, $(\Phi, \vec{A})$, is gauge dependent. The equations of motion (9), depend on the fields and are obviously gauge independent. However the change in the Lagrangian on a gauge transformation is represented by addition of a term which is total time derivative of a function of $(\vec{x}, t)$, and addition of a total time derivative does not alter the action integral or the equations of motion.

### *18.5 Manifestly covariant treatment*

To give a manifestly covariant description of the Lagrangian and Hamiltonian formalism, the customary variables $\vec{x}$ and $\vec{u}$ must be replaced by their four-vector counterparts $x^\alpha = (ct, \vec{x})$ and $U^\alpha = (\gamma c, \gamma \vec{u})$. The free-particle Lagrangian (7) can be written in terms of $U^\alpha$ as

$$L_{free} = -\frac{mc}{\gamma}\sqrt{U^\alpha U_\alpha} \tag{26}$$

Then the action integral (3) would be

$$A = -mc\int_{\tau_1}^{\tau_2}\sqrt{U^\alpha U_\alpha}\, d\tau \tag{27}$$

This is the manifestly invariant form of the action integral and should normally lead to the equations of motion. However there is one problem and that is that out of the four components of the four-velocity only three are really independent because of the constraint equation

$$U^\alpha U_\alpha = c^2 \tag{28}$$

or equivalently

$$U^\alpha \frac{dU_\alpha}{d\tau} = 0 \tag{29}$$

This has to be incorporated into the equations of motion. We cannot freely vary this action to find the equations of motion. Now the integrand in equation (27) is

$$\sqrt{U^\alpha U_\alpha}\, d\tau = \sqrt{g^{\alpha\beta}U_\alpha U_\beta}\, d\tau = \sqrt{g^{\alpha\beta}\frac{dx_\alpha}{d\tau}\frac{dx_\beta}{d\tau}}\, d\tau = \sqrt{g^{\alpha\beta}dx_\alpha dx_\beta}$$

This is the infinitesimal length element in four-space. This suggests that the action integral (27) be replaced by

$$A = -mc\int_{\tau_1}^{\tau_2}\sqrt{g^{\alpha\beta}\frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}}\, ds \tag{30}$$

$x^\alpha(s)$ is the four-vector coordinate of the particle, $s$ is a parameter that is monotonically increasing function of $\tau$, but otherwise arbitrary. The action integral is an integral along the world line of the particle, and the principle of least action is the statement that the actual path is the longest path, viz., the geodesic. We now treat each $\frac{dx^\alpha}{ds}$ as an independent generalized velocity, so the Lagrangian takes the functional form $L(x^\alpha, \frac{dx^\alpha}{ds}, s)$. After the calculus of variations has been completed we identify

$$\sqrt{g^{\alpha\beta} \frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}}\, ds = c\, d\tau \qquad (31)$$

and so impose the constraint (28). A straightforward variational calculation with equation (30) yields the Euler-Lagrange equations

$$mc\frac{d}{ds}\left[\frac{dx^\alpha / ds}{\sqrt{\frac{dx^\beta}{ds}\frac{dx_\beta}{ds}}} ds\right] = 0$$

Now imposing the constraint, we have

$$m\frac{d^2 x^\alpha}{d\tau^2} = 0$$

as expected for free particle motion.

For a charged particle in an external electromagnetic field the form of the Lagrangian (14) suggests that the manifestly covariant form of the action integral is

$$A = -\int_{\tau_1}^{\tau_2} [mc\sqrt{g^{\alpha\beta} \frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}} + q\frac{dx^\alpha}{ds} A^\alpha(x)]ds \qquad (32)$$

The Euler-Lagrange equations take the form

$$\frac{d}{ds}\left(\frac{\partial \tilde{L}}{\partial(\frac{dx_\alpha}{ds})}\right) - \partial^\alpha \tilde{L} = 0 \qquad (33)$$

The Lagrangian $\tilde{L}$ is

$$\tilde{L} = -[mc\sqrt{g^{\alpha\beta} \frac{dx_\alpha}{ds}\frac{dx_\beta}{ds}} + q\frac{dx_\alpha}{ds} A^\alpha(x)] \qquad (34)$$

Let us find the equations of motion of the charged particle from this Lagrangian. The derivative of $\tilde{L}$ with respect to $\frac{dx_\alpha}{ds}$ is

$$\frac{\partial \tilde{L}}{\partial(\frac{dx_\alpha}{ds})} = -\frac{\partial}{\partial(\frac{dx_\alpha}{ds})}[mc\sqrt{g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds}} + q\frac{dx_\gamma}{ds}A^\gamma(x)]$$

$$= -\frac{1}{2}mc(g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds})^{-1/2}\frac{\partial}{\partial(\frac{dx_\alpha}{ds})}[g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds}] - q\frac{\partial}{\partial(\frac{dx_\alpha}{ds})}[\frac{dx_\gamma}{ds}A^\gamma(x)]$$

We have changed the dummy indices in $\tilde{L}$ from $(\alpha,\beta) \to (\beta,\gamma)$, since an index cannot be used more than twice. Now

$$\frac{\partial}{\partial(\frac{dx_\alpha}{ds})}[g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds}] = g^{\beta\gamma}[\frac{dx_\gamma}{ds}\delta^\alpha{}_\beta + \frac{dx_\beta}{ds}\delta^\alpha{}_\gamma] = g^{\alpha\gamma}\frac{dx_\gamma}{ds} + g^{\beta\alpha}\frac{dx_\beta}{ds} = 2\frac{dx^\alpha}{ds}$$

$$\frac{\partial}{\partial(\frac{dx_\alpha}{ds})}[\frac{dx_\gamma}{ds}A^\gamma(x)] = \frac{\partial}{\partial(\frac{dx_\alpha}{ds})}[\frac{dx_\gamma}{ds}]A^\gamma(x) = \delta^\alpha{}_\gamma A^\gamma(x) = A^\alpha(x)$$

Hence

$$\frac{\partial \tilde{L}}{\partial(\frac{dx_\alpha}{ds})} = -mc(g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds})^{-1/2}\frac{dx^\alpha}{ds} - qA^\alpha(x)$$

and

$$\frac{d}{ds}\left(\frac{\partial\widetilde{L}}{\partial(\frac{dx_\alpha}{ds})}\right)=\frac{d}{ds}[-mc(g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds})^{-1/2}\frac{dx^\alpha}{ds}-qA^\alpha(x)]$$

$$=-mc\frac{d}{ds}[(g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds})^{-1/2}]\frac{dx^\alpha}{ds}$$

$$-mc[(g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds})^{-1/2}]\frac{d}{ds}(\frac{dx^\alpha}{ds})-q\frac{d}{ds}A^\alpha(x) \qquad (35)$$

$$=-mc\frac{d}{ds}[(g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds})^{-1/2}]\frac{dx^\alpha}{ds}$$

$$-mc[(g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds})^{-1/2}]\frac{d^2x^\alpha}{ds^2}]-q\frac{\partial A^\alpha}{\partial x_\gamma}\frac{dx_\gamma}{ds}$$

The second term of equation (33) is

$$\partial^\alpha\widetilde{L}=-\partial^\alpha[mc\sqrt{g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds}}+q\frac{dx_\gamma}{ds}A^\gamma(x)]=-q\frac{dx_\gamma}{ds}\partial^\alpha A^\gamma(x) \qquad (36)$$

Now at the end of the calculation we apply the constraint, equation (31)

$$(g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds})^{1/2}=c\,,$$

and replace $s$ by $\tau$. As a result [in equation (35)] we have

$$\frac{d}{ds}[(g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds})^{-1/2}]=0\,; \qquad (g^{\beta\gamma}\frac{dx_\beta}{ds}\frac{dx_\gamma}{ds})^{-1/2}=1/c$$

Substituting equation (36) and (35) with these changes into (33), we have

$$-m\frac{d^2x^\alpha}{d\tau^2}-\frac{q}{c}\frac{\partial A^\alpha}{\partial x_\gamma}\frac{dx_\gamma}{d\tau}+q\frac{dx_\gamma}{d\tau}\partial^\alpha A^\gamma(x)=0$$

or

$$m\frac{d^2x^\alpha}{d\tau^2}=q(\partial^\alpha A^\beta-\partial^\beta A^\alpha)\frac{dx_\beta}{d\tau}=qF^{\alpha\beta}\frac{dx_\beta}{d\tau} \qquad (37)$$

which is the covariant form of the equation of motion.

## **The Hamiltonian**

The *conjugate momentum four-vector* is defined by

$$P^{\alpha} = -\frac{\partial\tilde{L}}{\partial(\frac{dx_{\alpha}}{ds})} = mU^{\alpha} + qA^{\alpha} \qquad (38)$$

The minus sign in equation (38) is introduced so that it conforms to equation (19). A Hamiltonian can now be defined by

$$\tilde{H} = \frac{1}{2}(P_{\alpha}U^{\alpha} + \tilde{L}) \qquad (39)$$

Elimination of $U^{\alpha}$ by means of equation (36) leads to the expression

$$\tilde{H} = \frac{1}{2m}(P_{\alpha} - qA_{\alpha})(P^{\alpha} - qA^{\alpha}) - \frac{1}{2}mc^2 \qquad (40)$$

The Hamilton's equations are

$$\frac{dx^{\alpha}}{d\tau} = \frac{\partial\tilde{H}}{\partial P_{\alpha}} = \frac{1}{m}(P^{\alpha} - qA^{\alpha}) \qquad (41)$$

and

$$\frac{dP^{\alpha}}{d\tau} = -\frac{\partial\tilde{H}}{\partial x_{\alpha}} = \frac{q}{mc}(P_{\beta} - qA_{\beta})\partial^{\alpha}A^{\beta} \qquad (42)$$

We can easily show that these equations are equivalent to the Euler-Lagrange equations (37).

While the above Hamiltonian is formally satisfactory, it has several problems. We have seen that the Hamiltonian is the time component of a four-vector. However, the Hamiltonian of equation (40) is a Lorentz scalar, not an energy-like quantity. Furthermore, use of equation (28) and (38) shows that $\tilde{H} \equiv 0$. Clearly such a Hamiltonian formulation differs considerably from the familiar non-relativistic version.

***Summary***

1. *This module is devoted to the Lagrangian and Hamiltonian formalism for a charged particle in the presence of electromagnetic fields. The idea of Lagrangian dynamics in non-relativistic classical mechanics is reviewed.*
2. *The Lagrangian for the relativistic motion of a free particle is derived and then extended to a charged particle in the presence of an electromagnetic field.*
3. *Next the relativistic charged particle in the presence of an electromagnetic field is described in a Hamiltonian formulation and the Hamiltonian of the particle obtained.*
4. *The treatment so far is relativistic but non-covariant. The theory is then reformulated in an explicitly covariant form.*